Strain 2. The Mohr circle for strain

Calculating extension and rotation

Let's consider pure shear deformation with the FSAs parallel to the coordinate axes, as shown in figure 1. For each unit vector with angle α relative to e_1, we will now determine the extension and rotation as a function of the amount of strain and α.

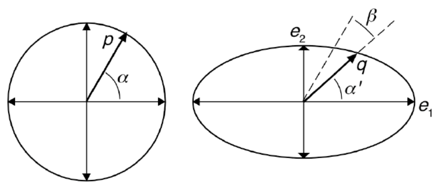


Figure 1. Undeformed and deformed state, with finite stretching axes parallel to the x- and y-axis. A vector $\underline{p}$ oriented at an angle α deforms to a vector $\underline{q}$ at angle α' to e_1. The vector has rotated an angle β.

With the conditions above, we can write the simplified equations that relate the undeformed unit vector ($\underline{p}$) to the deformed one ($\underline{q}$):

$$\begin{aligned} q_x &= e_1 p_x = e_1 \cos(\alpha) \\ q_y &= e_2 p_y = e_2 \sin(\alpha) \end{aligned} \tag{1}$$

The length of the unit vector p is by definition one, which means that the length of the deformed vector q is equal to its extension: $e_{(\alpha)}$.
The angle β between $\underline{p}$ and $\underline{q}$ can be determined with the dot product or cross product:

$$|q|\cos(\beta) = \underline{p}\cdot\underline{q} \Leftrightarrow e_{(\alpha)}\cos(\beta) = \cos(\alpha)e_1\cos(\alpha) + \sin(\alpha)e_2\sin(\alpha) \tag{2}$$

$$|q|\sin(\beta) = \underline{q}\times\underline{p} \Leftrightarrow e_{(\alpha)}\sin(\beta) = e_1\cos(\alpha)\sin(\alpha) - e_2\sin(\alpha)\cos(\alpha) \tag{3}$$

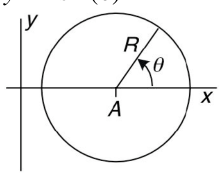
These two equations should hopefully look familiar. If not, check the equations for the stress tensor in the lecture notes "Structural Geology", which are of exactly the same type. There we showed that these equations actually describe a circle if we rewrite them the right way:

$$e_{(\alpha)}\cos(\beta) = \frac{e_1 + e_2}{2} + \frac{e_1 - e_2}{2}\cos(2\alpha) \tag{4}$$

$$e_{(\alpha)}\sin(\beta) = \frac{e_1 - e_2}{2}\sin(2\alpha) \tag{5}$$

The Mohr circle for finite strain

Equation for a circle with its centre on the x-axis:
$x = A + R\cos(\theta)$
$y = R\sin(\theta)$



Exactly the same way as for stress, we have now derived equations that describe the strain with a Mohr circle:

Centre of circle: $A = \dfrac{e_1 + e_2}{2}$ (6a)

Radius of circle: $R = \dfrac{e_1 - e_2}{2}$ (6b)

Angle to use: $\theta = 2\alpha$ (6c)

x-axis: $x = e_{(\alpha)}\cos(\beta)$ (6d)

y-axis: $y = e_{(\alpha)}\sin(\beta)$ (6e)

With these definitions we can draw the *Mohr circle for finite strain* (Fig. 2).

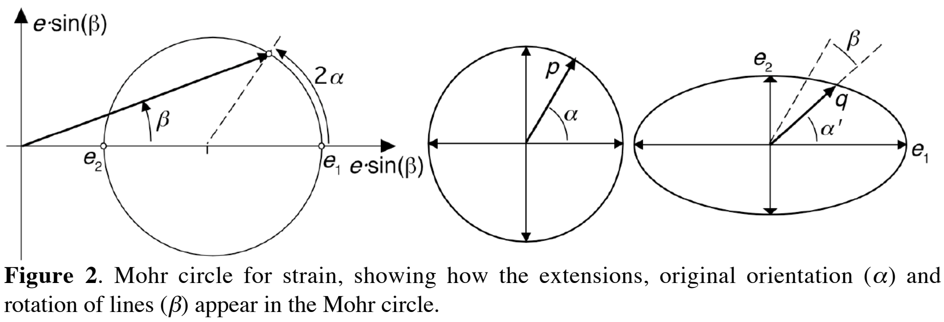


Figure 2. Mohr circle for strain, showing how the extensions, original orientation (α) and rotation of lines (β) appear in the Mohr circle.

It should be stressed here that the Mohr circle construction used here was derived for a pure shear deformation with the FSA's parallel to the x- and y-axis of our reference frame: lines parallel to these axes do not rotate. The position gradient tensor for such a deformation is symmetric as the values away from the diagonal are all zero. We have seen before that this is not always the case for finite strain. In case of simple shear, for instance, we have a non-zero value away from the diagonal. How to deal with this is explained below.

The Mohr circle for general strain

Figure 3 shows two deformations of one and the same *Asaphus* trilobite. In the one case we have simple shear deformation and in the other case pure shear. Both deformations lead to the same finite shape of the deformed trilobite. The first lesson we learn from this is that we cannot simply discern pure and simple shear from the finite shape of an object or the strain ellipse alone. Any deformation type can lead to the same fine shape. It all depends on the *orientation* of the object that is deformed *relative* to the applied deformation. The absolute amount of deformation, expressed in the ratio of the finite stretching axes (R_f), must however be the same.

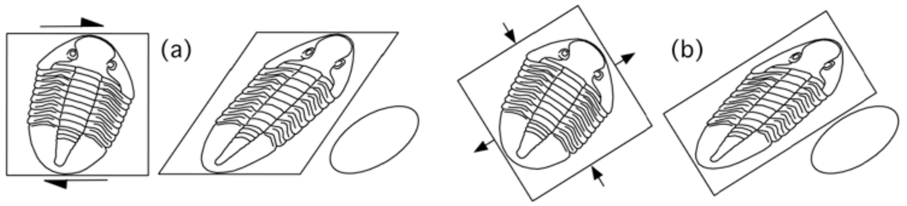


Figure 3. Deformation of an *Asaphus* trilobite. (a) Oblique simple shear deformation to a shear strain of γ=0.69, resulting in a finite strain ratio (R_f) of 2.02 (see strain ellipse). (b) Pure shear deformation, also to R_f=2.02. Notice that both deformations lead to exactly the same finite shape of the trilobite.

Figure 4 shows the deformation of the same trilobite as figure 3, but to a dextral shear strain of γ=1, parallel to the horizontal *x*-axis (Fig. 4b). The finite strain ratio (R_f) is 2.62. Because simple shear deformation is non-coaxial, the material lines that form the finite stretching axes (FSAs) rotate. The line of maximum extension (e_1=1.618) rotated by β=26.6°, from ζ=58.3° to ζ'=31.7°. In figure 4c we applied a pure shear deformation, also to a finite strain ratio of 2.62. For the pure shear deformation we used a reference frame (X,Y) at ζ=58.3° to the x,y reference frame, with maximum elongation parallel to the *X*-axis. Because the FSAs in pure shear do not rotate (coaxial deformation), the finite strain ellipse has its long axis still oriented parallel to the *X*-axis, or at ζ=58.3° to the *x*-axis.

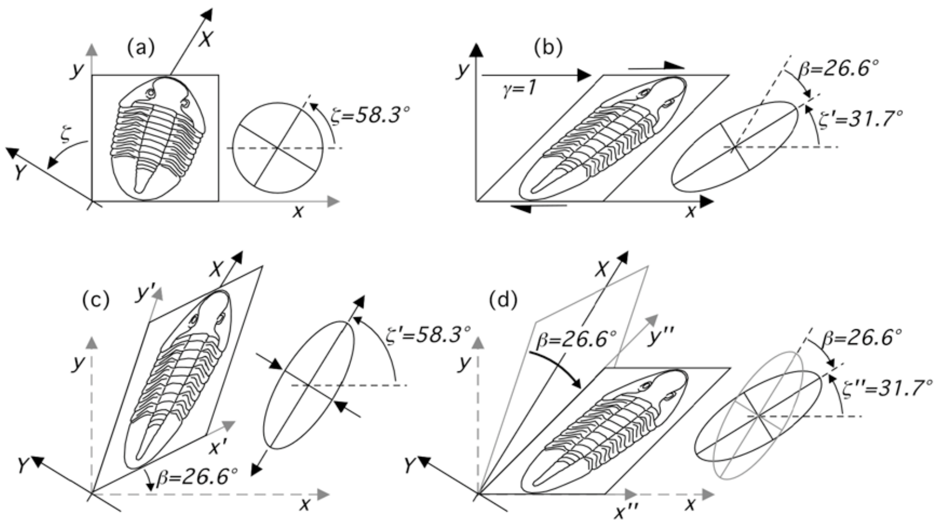


Figure 4. (a) Undeformed state of an *Asaphus* trilobite, together with a circle that represents the undeformed strain ellipse. One reference frame (x,y) is parallel to the box containing *Asaphus*. The other reference frame (X,Y) makes an angle of ζ=58.8° with the x,y-frame. This reference frame is parallel to those material lines that will become the FSAs. (b) State after a simple shear deformation with γ=1 parallel to the *x*-axis. The strain ellipse has an axial ratio of R_f=2.62. (c) State after pure shear deformation with R_f=2.62 with principal stretching axes parallel to the *X*- and *Y*-axis. Because of the pure shear deformation, the material lines that make the FSA's did not rotate. (d) State after a right hand rigid-body rotation of β=26.6° of the pure shear deformed *Asaphus* shown in (c). Notice that the situation in (d) is exactly the same as in (b).

In the previous section we have seen how to construct a Mohr circle for finite strain for pure shear with the principal stretching axis parallel to the axes of our reference frame. This means we can draw the Mohr circle for the pure shear deformation in our *X,Y*-reference frame (Fig. 5). Our two principal extensions are e_1=1.62 and e_2=0.62. The Mohr circle has a radius of one in this case.

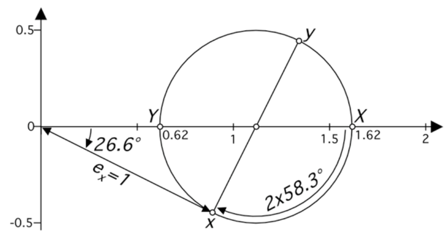


Figure 5. Mohr circle for the deformation shown in figure 4b. FSAs are parallel to the *X*- and *Y*-axes. The horizontal *x*-axis makes an angle of 58.8° with the *X*-axis. Using double angles in the Mohr circle, we can find the position of the *x*-axis in the circle. The extension (e_x) of the *x*-axis (distance from origin in diagram) is exactly one. The *x*-axis has rotated to the left by 26.6°.

The Mohr circle tells us the extension of any line and its rotation. The *x*-axis made an angle of 58.3° with the *X*-axis, so we can now measure double that angle to find the extension and rotation of the *x*-axis. Because of our definition of the sign of angles (Fig. 2), we measure 2×58.8° clockwise from the point representing the *X*-axis. This point lies at a distance of exactly one from the origin, meaning that the *x*-axis neither stretched nor shortened (e_x=1). It has rotated 26.6° anti-clockwise, as we can see in figure 4c.

Adding rigid-body rotation to the Mohr circle for strain

In figure 4d we added a right-hand, 26.6° rigid-body rotation to the pure-shear deformed *Asaphus*. The result is now exactly the same as that of simple shear deformation parallel to the *x*-axis (Fig. 4b). This means that simple shear finite strain can be described by a summation of a pure shear strain and a rigid-body rotation. This can be generalised to an important observation:

> ***"Any general finite strain can be described by a combination of a pure shear component and a rigid-body rotation component"***

Now don't think that this means that there is no difference between pure shear and other types of deformation, because there is a difference! However, if we only compare the undeformed state and the finite deformed state, we may not always see the difference. This is the case here. Later we will have a closer look at the meaning of simple shear, pure shear and general shear.

Rigid-body rotation

Rigid-body rotation means that we rotate all lines by exactly the same angle, without stretching or shortening any line. We can do this in a Mohr diagram. The distance from the origin of a point on the Mohr circle represents its extension. The angle the line from the origin to that point makes with the horizontal axis in the diagram is the angle that the line, represented by that point, rotated. To add a rotation, without any stretching or shortening, then just means rotating the Mohr diagram around the origin. All points on the circle then get a certain rotation angle added or subtracted, but their elongations remain unchanged. This is done in figure 6, where we rotated the diagram by 26.6°, which puts the point representing the *x*-axis on the horizontal axis, at a unit distance from the origin. This means the *x*-axis did not stretch and did not rotate. Because the deformed state in figure 4b is the same as in figure 4d, figure 6 actually shows the Mohr circle for dextral simple shear parallel to the *x*-axis. The shear strain is one, meaning the *y*-axis rotated by tan(1) = 45°.

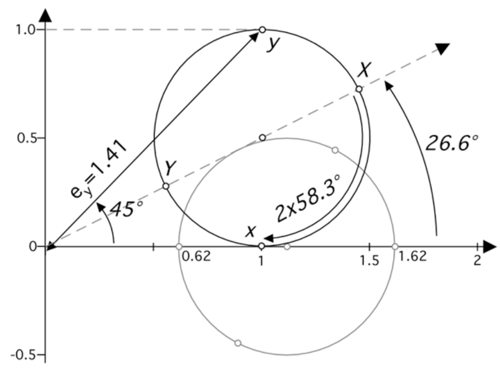


Figure 6. Mohr circle for dextral simple shear parallel to *x*-axis. Constructed by rotating the diagram of figure 5 by 26.6°. The *x*-axis now lies on the horizontal axis, at a distance one from the origin. The *y*-axis lies at a distance of 1.41 from the origin, meaning it stretched by 41% and it rotated 45°.

Vorticity

The Mohr circle for simple shear does not lie centred on the horizontal axis. In fact, all points on the circle lie on (*x*-axis) or above the horizontal, meaning all lines rotated in the same direction. The average rotation of all material lines is given by the line that goes through the centre of the circle, here 26.6°. This is of course the same as the rigid body rotation we added after our pure shear step. In pure shear, the average rotation is 0°, because just as many lines rotate to the left as to the right. If we then add a constant rotation angle to all lines, naturally we get that the average rotation is exactly that angle (26.6° here). This average rotation we call the *vorticity*.

Vorticity

Clearly, the position of the Mohr circle relative to the horizontal axis in the diagram tells us something about the vorticity of deformation. If the circle is centred on the horizontal axis, vorticity is zero and we have pure shear. If the circle is not centred on the horizontal axis, average rotation is not zero and vorticity is non-zero. This is the case for general shear. If the circle just touches the horizontal (as in Fig. 6) we have the special case of simple shear.

Size of the Mohr circle

Maximum and minimum extension (e_1 and e_2 respectively) are given by the points furthest and closest to the origin. These points lie on the line through the origin and the centre of the circle (marked with *X* and *Y* in figure 6). The bigger the circle, the bigger the difference between maximum and minimum elongation, and hence the absolute finite strain or finite strain ratio (R_f=e_1/e_2).

The Mohr circle and the position gradient tensor

If you know the position gradient tensor (eq. 20), you can actually immediately draw the Mohr circle for finite strain. The position gradient tensor in the *X*-*Y*-reference frame for the pure shear deformation shown in figures 4c and 6 is:

$$\mathbf{F}_{ps} = \begin{pmatrix} 1.62 & 0 \\ 0 & 0.62 \end{pmatrix}$$

This means that a vector with coordinates {1,0} (unit vector parallel to *X*-axis) deforms to {1.62,0}. The perpendicular unit vector {0,1} (parallel to the *Y*-axis) deforms to {0,0.62}. Now look at the coordinates in the Mohr diagram (Fig. 5), the points representing the *X*- and *Y*-axis get: {0,1.62} and {0,0.62}. The elongation values 1.62 and 0.62 appear in both the position gradient tensor and in the coordinates of the reference frame axes in the Mohr diagram.

Now we do the same for simple shear deformation (Figs. 4b and 6), which has a shear strain of one, giving a position gradient tensor (in our *x,y*-reference frame):

$$\mathbf{F}_{ss} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Vector {0,1} now deforms to {0,1} and has coordinates {0,1} in the Mohr diagram. Vector {1,0} deforms to {1,1} and has coordinates {1,1} in the diagram. It looks like we can use the coordinates of the deformed unit vectors parallel to our reference frame axes to construct our Mohr circle.

Let's now consider a general 2-dimensional position gradient tensor:

$$\mathbf{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

Vector $\{1,0\}$ deforms to $\{1\times a + 0\times b, 1\times c + 0\times d\} = \{a,c\}$ (8a)

Vector $\{0,1\}$ deforms to $\{0\times a + 1\times b, 0\times c + 1\times d\} = \{d,b\}$ (8b)

It now turns out that the coordinates of the points representing the reference frame axes are in fact $\{a,-c\}$ for the x-axis and $\{b,d\}$ for the y-axis. Check for yourself if this is true for the example of the deformed trilobite above.

The axes of our reference frame make an angle of 90° with each other, meaning that they should lie at 180° from each other, measured along the outline of the Mohr circle (double angles!). This means that the two points should lie opposite to each other on the circle. This also means that when we have the points, we can draw a circle, with its origin halfway between the two points.

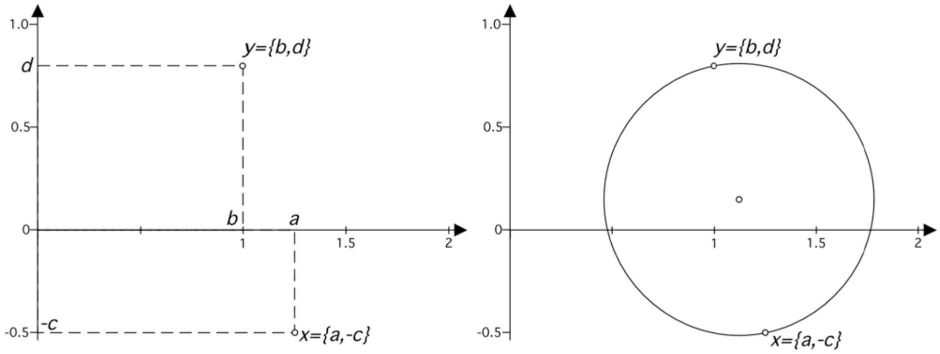
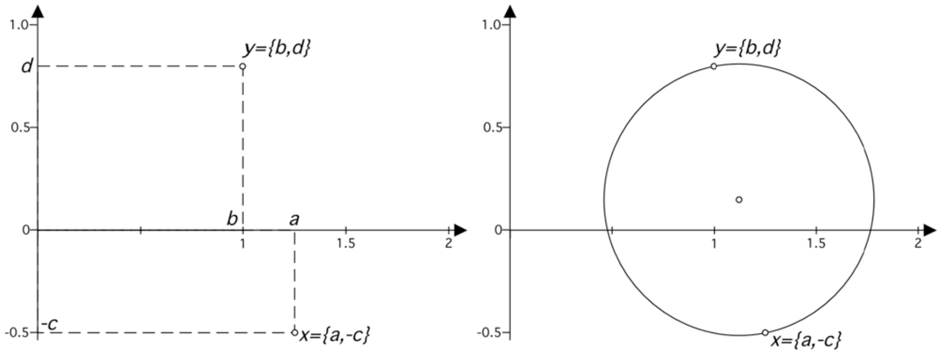


Figure 7. If you know the position gradient tensor (a,b,c,d), you can draw the Mohr circle for finite strain. First plot the x- and y-axis with coordinates $\{a,-c\}$ and $\{d,b\}$ in the diagram. Then you can construct the circle, using the fact that the circle must go through the two points and that these point lie opposite each other on the circle.

Figure 7. If you know the position gradient tensor (a,b,c,d), you can draw the Mohr circle for finite strain. First plot the x- and y-axis with coordinates $\{a,-c\}$ and $\{d,b\}$ in the diagram. Then you can construct the circle, using the fact that the circle must go through the two points and that these point lie opposite each other on the circle.



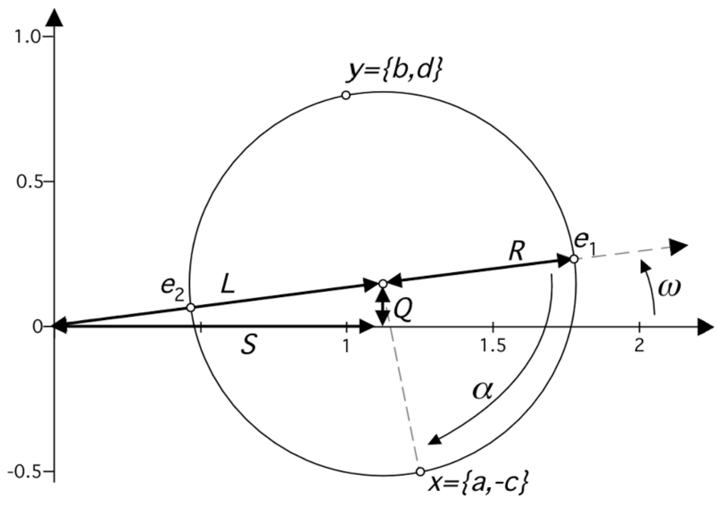


Figure 8. Mohr circle for finite strain for a general finite strain case, defined by the four values of the position gradient tensor (a,b,c,d).

The general Mohr circle for finite strain

Figure 8 shows a Mohr circle for a general finite strain case with a position gradient tensor as defined in equation 8. As described above, the circle can be constructed if the position gradient is known. The radius R of the circle is given by:

$$R = \tfrac{1}{2}\sqrt{(a-b)^2 + (d+c)^2} \tag{30}$$

The centre of the circle has coordinates $\{S,Q\}$, with:

$$S = \tfrac{1}{2}(a+b) \tag{31}$$

$$Q = \tfrac{1}{2}(d-c) \tag{32}$$

L is the distance from the origin of the graph to the centre of the circle:

$$L = \sqrt{S^2 + Q^2} \tag{33}$$

Kinematic vorticity number

We have seen before that the hight of the centre of the circle (Q) is the vorticity, which is the average rotation of all material lines. The vorticity depends on the type of deformation, but also on the amount of deformation. The latter is determined by the size of the circle (defined by the radius R). It is therefore useful to normalise the vorticity to the amount of strain. We can do this by defining the *kinematic vorticity number*, W_k, defined as:

$$W_k = \left|\frac{Q}{R}\right| \tag{34}$$

The kinematic vorticity number is positive or negative, depending on the direction of the shear component. There are some special cases:

- $W_k = 0$ (Q=0): Pure shear – circle centred on horizontal axis;
- $W_k = \pm 1$ (Q=R): Simple shear – circle just touches horizontal axis;
- $W_k = \infty$ (R=0): Rigid body rotation only – circle is a point.

General shear

The example of figure 8 shows a case of W_k between 0 and 1, which is a case of *general shear* between pure and simple shear. The finite strain ration (R_f) can also be determined directly from the Mohr circle:

$$R_f = \frac{e_1}{e_2} = \frac{L+R}{L-R} \tag{34}$$

Dilation

We may be interested to know whether we had an area change or dilation (ΔA). The area change is the product of e_1 and e_2, which can be determined from the circle:

$$\Delta A = e_1 \cdot e_2 = (L+R)(L-R) = L^2 - 2LR + R^2 \tag{35}$$

If $\Delta A>1$, we had an increase in area, and if $\Delta A<1$ we had a decrease in area. Such change in area may occur by material gain or loss, due to, for example, compaction or dissolution and precipitation of material. However, often we see dilation in one plane because of extension or shortening in the third dimension.